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Wave-Absorbing Control for Flexible Structures with Noncollocated Sensors and Actuators

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A control method for vibration suppression of flexible structures with noncollocated sensors and actuators is introduced. A simple expression of response of flexible structural networks is also shown. The control method is an extension of wave-absorbing control based on the concept of disturbance propagation. Disturbance propagation in a flexible structure modeled as a distributed parameter system is treated in the frequency domain, and the usual modal expansions are not employed for modeling the system. Response of structural networks is obtained from exact solutions of Laplace-transformed equations of motion. Boundary conditions are expressed in terms of reflection/transmission of traveling waves, and controllers with noncollocated sensors and actuators are designed to suppress outgoing waves at boundaries of the structure. Responses of the controlled systems to external disturbances are analyzed through numerical examples.

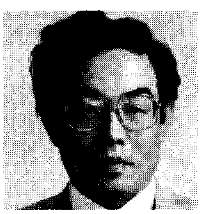
I. Introduction

SUPPRESSION of traveling disturbances of flexible space structures becomes one of significant topics in the control problem of large space structures (LSS) in accordance with increase of size and flexibility of LSS. Disturbances applied on an LSS propagate over the structure through structural members, reflect at ends or junctions of the structure, and finally form standing waves. Control of traveling waves is required to cancel the generated disturbances in the neighborhood of each excited point before they propagate over the structure.

The conventional modal approach to the vibration control is also applicable to control of traveling waves. Modal control of traveling waves, however, needs a large number of modes to describe traveling waves in a sufficient degree of accuracy and results in a controller that requires much computation. Benighof and Meirovitch[1] apply independent modal-space control (IMSC) and direct feedback control to active suppression of traveling waves in structures. Their approach requires a number of actuators as well as controlled modes, and it is demonstrated that direct feedback control is more suitable than IMSC truncated to low order for the problem in which a large number of higher modes have to be controlled.

Wave-absorbing control, which results from a description of the structural response in terms of propagating elastic disturbances, is the alternative to the modal control method, and its theoretical features are more suitable for control of traveling waves than the modal approach. System model of the structures is treated as distributed parameter system in frequency domain, and the modal methods are not employed in the present analysis. The model responses are described in terms of propagating disturbances.

The concept of the wave-absorbing control is applied to the control of LSS by von Flotow[2,3] and von Flotow and Schäfer.[4] They introduce the viewpoint that the elastic response of large spacecraft structures may be aptly viewed in terms of the disturbance propagation characteristics of the structure[2] and



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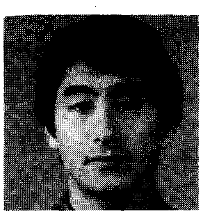

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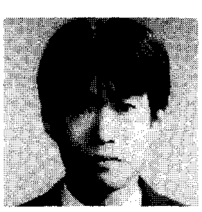

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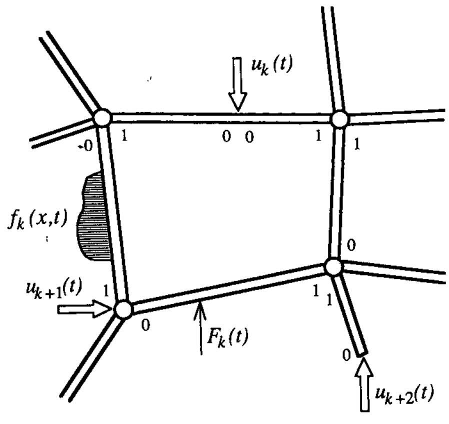


Fig. 1 Conceptual flexible structure.

apply the wave-absorbing control based upon such a concept to a laboratory structure.[3] Active vibration isolation by canceling traveling waves based on the same concept is studied in Refs. 5-9. The technique aims to isolate one part of a structure from disturbances excited elsewhere in the structure. Actuators are located at midpoints of structural members and cancel passing disturbances so that the disturbances do not disturb a downstream portion that should be isolated dynamically. Miller and von Flotow[10] and Miller et al.[11] apply the traveling wave approach to power flow in structural networks[10] and introduce optimal control of power flow at structural junctions.[11]

This paper discusses design procedures of wave-absorbing controllers with noncollocated sensors and actuators. Flexible structures are modeled as networks of one-dimensional media that transmit traveling waves. External control forces are applied at junctions to control boundary conditions of governing partial differential equations of the system. Formulation of structural dynamics proposed in this paper enables simple expression of global dynamics as well as local dynamics in the sense that there is no difference in the expression between a case with multiple members and a case with a single member. The wave-absorbing controller is one of low authority controllers (LAC)[12] and acts mainly to augment damping of structures as well as the direct velocity feedback (DVFB) control.[13] Only vibration control of structures is considered, being separated from control of rigid motion or the attitude motion of LSS. Therefore rigid motion of structures is not taken into consideration in the formulation of structural dynamics. A controller with noncollocated sensors and actuators (noncollocated controller) has many favorable features for implementation from a practical point of view. This paper shows the advantages of describing the model dynamics in terms of propagating disturbances; however, optimization of sensor/actuator location is beyond the scope of the present paper.

II. Dynamics of Structural Networks

This section gives a simple expression of response of structural networks in the frequency domain. The response is expressed in terms of either cross-sectional states of members or traveling waves in members. The rigid motion of structures is not taken into consideration in the formulation of structural dynamics. The approach taken here to derive expression of structural response is similar to the approach employed in a few works concerned with interconnections of lumped and distributed parameter subsystems[14,15] making use of exact solutions of Laplace-transformed equations of motion, although the models are different. It is assumed that structural networks treated here consist of one-dimensional members, and each member is uniform. A conceptual structure that consists of one-dimensional flexible members is illustrated in Fig. 1. Excited disturbances on such a structure propagate along the members in the form of longitudinal, lateral, or torsional vibration. It is assumed that the behavior of the structure is governed by a set of partial differential equations[16]

$$L_i\left[y_i(x,t)\right] + \frac{\partial}{\partial t} C_i\left[y_i(x,t)\right] + m_i \frac{\partial^2 y_i(x,t)}{\partial t^2} = f_i(x,t) + \sum_j F_{ij}(t)\delta(x-x_j), \qquad (0<x<l_i,\ i=1,2,\ldots,n_0) \quad (1)$$

where $y_i(x,t)$ are displacements, n_0 is the number of all kind of displacements to be considered in the analysis, and l_i denotes the length of each member. The linear homogeneous self-adjoint differential operators L_i and C_i consist of derivatives through order $2p_i$ with respect to the spatial coordinate x. The spatial coordinate x is taken along each member from one end ($x=0$) to the another end ($x=l_i$). The mass per unit length m_i is assumed constant along the length, and the excitation consists of the distributed force $f_i(x,t)$ and the concentrated forces of amplitude $F_{ij}(t)$ acting at points $x=x_j$. The symbol $\delta(x-x_j)$ denotes a spatial Dirac's delta function. The partial differential equations are coupled with one another by $2n$ boundary conditions at the boundaries of the structure:

$$\sum_{j=1}^{n_0} B_{ij}\left[y_j(x,t)\right] = \sum_{j=1}^{m} G_{ij}\left[u_j(t)\right], \qquad (i=1,2,\ldots,2n) \quad (2)$$

$$n = \sum_{i=1}^{n_0} p_i \quad (3)$$

where x is equal to 0 or l_i, B_{ij} are linear homogeneous differential operators containing derivatives of order through $2p_i-1$, and G_{ij} are linear homogeneous operators that express transmission properties of control forces u_j ($i=1,2,\ldots,m$) to the boundaries. Only concentrated control forces are considered in this analysis, since distributed control forces in practical application are hard to implement. All concentrated control forces can be included without loss of generality in the boundary conditions and not in the forces in the right-hand side of Eq. (1). A member that has an actuator at a midpoint can be regarded as two members that are interconnected at the actuator location.

The Laplace transform of Eqs. (1) and (2) with respect to time yields n_0 ordinary differential equations with respect to the spatial variable x, in which the Laplace transform variable s is regarded as a parameter. Order of each ordinary differential equation is equal to $2p_i$ ($i=1,2,\ldots,n_0$). We employ the same notation for the transformed variables as their time-dependent equivalents to avoid new symbols. By introducing state vectors

$$y_i(x,s) = \left[\alpha_{i1} y_i(x,s) \quad \alpha_{i2}\frac{\mathrm{d}y_i(x,s)}{\mathrm{d}x} \quad \cdots \quad \alpha_{i2p_i}\frac{\mathrm{d}y_i^{(2p_i-1)}(x,s)}{\mathrm{d}x}\right]^T \qquad (i=1,2,\ldots,n_0) \quad (4)$$

in which α_{ij} are constants, each ordinary differential equation is expressed as a first-order ordinary differential equation in the matrix form:

$$\frac{\mathrm{d}y_i(x,s)}{\mathrm{d}x} = A_i(s)y_i(x,s) + f_i(x,s), \quad (y_i, f_i \in C^{2p_i}) \quad (5)$$

where the vector $f_i(x,s)$ consists not only of external forces in the right side of Eq. (1) but also of such initial conditions of Eq. (1) as $y_i(x,0)$ and $\partial y_i(x,0)/\partial t$, and usually has a sparse structure with only the $2p_i$th entry nonzero. Entries of the state vector $y_i(x,s)$ represent the Laplace transforms of such cross-sectional states of members as the axial displacement, the transverse displacement, the angle of twist, the slope, or the bending moment at the point x. The response of the whole system is described in the same manner as Eq. (5) by assembling equations for members into one equation. The spatial coordinate $x(0 \le x \le l_i)$ is replaced by the dimensionless form $l_i x\ (0 \le x \le 1)$ in the assembled description. This replacement is convenient to express the whole Laplace-transformed boundary conditions in a simple matrix form. Finally, the first-order ordinary differential equation that describes the response of the whole system is obtained in the matrix form

$$\frac{\mathrm{d}y(x,s)}{\mathrm{d}x} = A(s)y(x,s) + f(x,s), \quad (0<x<1;\ y,f \in C^{2n}) \tag{6}$$

subject to the boundary condition

$$B_0(s)y(0,s) + B_1(s)y(1,s) = B(s)u(s), \quad (u \in C^m) \tag{7}$$

where

$$y(x,s) = \left[y_1^T(x,s)\ \ y_2^T(x,s)\ \ \cdots\ \ y_{n_0}^T(x,s)\right]^T \tag{8a}$$

$$f(x,s) = \left[f_1^T(x,s)\ \ f_2^T(x,s)\ \ \cdots\ \ f_{n_0}^T(x,s)\right]^T \tag{8b}$$

$$A(s) = \text{block-diag}\left[A_1(s), A_2(s), \ldots, A_{n_0}(s)\right] \tag{8c}$$

The boundary condition in Eq. (7) is equivalent to the boundary conditions in the time domain, Eq. (2). It should be noted that Eq. (6) is an ordinary differential equation with respect to the spatial coordinate, and the state vector is of finite dimensions $2n$. These features are different from those of usual state-space expression of distributed parameter systems, which is an ordinary differential equation with respect to time and requires a state vector of infinite dimensions. Integration of Eq. (6) yields the relationship between cross-sectional state vectors at two points as

$$y(x,s) = \phi(x,x_0,s)y(x_0,s) + y_f(x,x_0,s) \tag{9}$$

where $\phi(x,x_0,s)$ denotes the transfer matrix, and $y_f(x,x_0,s)$ represents contribution of external forces:

$$\phi(x,x_0,s) = e^{(x-x_0)A(s)} \tag{10a}$$

$$y_f(x,x_0,s) = \int_{x_0}^{x} \phi(x,\eta,s)f(\eta,s)\,\mathrm{d}\eta \tag{10b}$$

The open-loop response of the structural network is obtained from Eqs. (7) and (9) as follows:

$$y(x,s) = \phi(x,0,s)\left[B_0(s) + B_1(s)\phi(1,0,s)\right]^{-1} \times \left[B(s)u(s) - B_1(s)y_f(1,0,s)\right] + y_f(x,0,s) \tag{11}$$

Next, propagation and reflection of traveling waves in the structural network is analyzed. Diagonalization of Eq. (6) transforms the cross-sectional state vector $y(x,s)$ to a new state vector $w(x,s)$, each element of which represents a traveling wave mode:

$$y(x,s) = Y(s)w(x,s), \quad (w \in C^{2n}) \tag{12}$$

$$\frac{\mathrm{d}w(x,s)}{\mathrm{d}x} = \Lambda(s)w(x,s) + Y^{-1}(s)f(x,s) \tag{13}$$

$$w(x,s) = \left[a(x,s)^T\ \ b(x,s)^T\right]^T, \quad (a,b \in C^n) \tag{14a}$$

$$\Lambda(s) = \text{diag}\left[\lambda_1(s), \ldots, \lambda_n(s),\ -\lambda_1(s), \ldots, -\lambda_n(s)\right] \tag{14b}$$

where $\lambda_i(s)$ and $-\lambda_i(s)$ $(i = 1,2,\ldots,n)$ are the eigenvalues of the matrix $A(s)$, and $Y(s)$ is a matrix of the corresponding eigenvectors. The eigenvalues of $A(s)$ appear in pairs $\lambda_i(s)$, $-\lambda_i(s)$ in accordance with pairs of wave modes traveling in opposite directions. The upper half $a(x,s)$ of $w(x,s)$ corresponds to the eigenvalues $\lambda_i(s)$ and represents wave modes traveling in the negative x direction. The lower half $b(x,s)$ corresponds to the eigenvalues $-\lambda_i(s)$ and represents wave modes traveling in the positive x direction. It is observed from Eqs. (12–14) that the cross-sectional state variables are synthesized from wave modes, and dynamics of each wave mode is independent of other wave modes. A wave mode in a certain member does not have interaction with cross-sectional state variables in other members; therefore the matrix $Y(s)$ has a sparse structure. Integration of Eq. (13) or transformation of Eq. (9) gives the relationship similar to Eq. (9) between wave modes at two points as follows:

$$w(x,s) = T(x,x_0,s)w(x_0,s) + w_f(x,x_0,s) \tag{15}$$

where

$$T(x,x_0,s) = e^{(x-x_0)\Lambda(s)} = \text{diag}\left[e^{(x-x_0)\lambda_1(s)}, \ldots, e^{(x-x_0)\lambda_n(s)}, e^{(x_0-x)\lambda_1(s)}, \ldots, e^{(x_0-x)\lambda_n(s)}\right] \tag{16a}$$

$$w_f(x,x_0,s) = \int_{x_0}^{x} T(x,\eta,s)Y^{-1}(s)f(\eta,s)\,\mathrm{d}\eta = Y^{-1}(s)y_f(x,x_0,s) \tag{16b}$$

Outgoing traveling waves at the boundaries consist of the wave modes $b(0,s)$ and $a(1,s)$, and incoming traveling waves are identical to the wave modes $a(0,s)$ and $b(1,s)$. The boundary condition, Eq. (7), is transformed to express generation of the outgoing waves $w_{\text{out}}(s)$ due to both of the incoming waves $w_{\text{in}}(s)$ and control forces $u(s)$ as follows:

$$w_{\text{out}}(s) = S(s)w_{\text{in}}(s) + R(s)u(s), \quad (w_{\text{out}}, w_{\text{in}} \in C^{2n}) \tag{17}$$

where

$$w_{\text{out}}(s) = \left[b^T(0,s)\ \ a^T(1,s)\right]^T$$

$$w_{\text{in}}(s) = \left[a^T(0,s)\ \ b^T(1,s)\right]^T \tag{18a}$$

$$S(s) = -B_{\text{out}}^{-1}(s)B_{\text{in}}(s), \qquad R(s) = B_{\text{out}}^{-1}(s)B(s) \tag{18b}$$

$$B_{\text{out}}(s) = B_0(s)Y(s)T_3 + B_1(s)Y(s)T_2$$

$$B_{\text{in}}(s) = B_0(s)Y(s)T_1 + B_1(s)Y(s)T_4 \tag{18c}$$

$$T_1 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \tag{18d}$$

In the previous expression, the matrix $S(s)$ is the open-loop scattering matrix, and I_n denotes an $n \times n$ unity matrix. Both matrices $B_{\text{out}}(s)$ and $B_{\text{in}}(s)$ are obtained by dividing and reordering $B_0(s)Y(s)$ and $B_1(s)Y(s)$, and the matrices T_i $(i = 1,2,3,4)$ perform those operations. Equation (17) is the basis for design of wave-absorbing controllers to cancel outgoing waves at boundaries.

III. Wave-Absorbing Controller Design

This section presents some design procedures of wave-absorbing controllers to cancel outgoing waves at boundaries

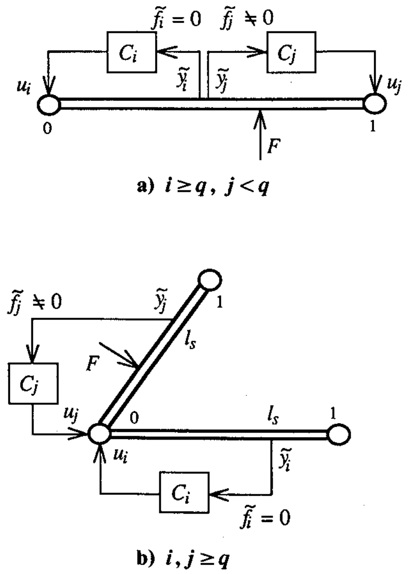


Fig. 2 The relation between the vector $\tilde{f}(s)$, sensor positions, and the parameter q.

being based on the relationship Eq. (17). We consider output feedback specified in the form as

$$u(s) = C(s)\tilde{y}(s), \qquad (\tilde{y} \in C^r) \tag{19}$$

where the matrix $C(s)$ represents a frequency-dependent gain matrix, and the vector $\tilde{y}(s)$ denotes sensor outputs that are linear combinations of the cross-sectional state variables at the sensing points in the structure. Such sensor outputs can be expressed in the following form using Eq. (9):

$$\tilde{y}(s) = \tilde{\phi}(s)y(0,s) + \tilde{y}_f(s), \qquad (\tilde{y}_f \in C^r) \tag{20}$$

where the matrix $\tilde{\phi}(s)$ has rows with linear combinations of rows of $\phi(x_i,0,s)$, and the vector $\tilde{y}_f(s)$ has entries with linear combinations of entries of $y_f(x_i,0,s)$, where x_i denotes sensor positions. The open-loop sensor output is obtained from Eqs. (11) and (20):

$$\tilde{y}(s) = \tilde{\phi}(s)\left[B_0(s) + B_1(s)\phi(1,0,s)\right]^{-1} \times \left[B(s)u(s) - B_1(s)y_f(1,0,s)\right] + \tilde{y}_f(s) \tag{21}$$

When the external disturbances are concentrated forces, a partial realization of Eq. (21) gives a truncated state-space model of the structural network and leads to the state-space approach for controller design. In contrast to the state-space approach, the traveling wave approach treated in this paper does not include the truncation procedure and uses the boundary condition Eq. (17), which is expressed in terms of the incoming and outgoing waves at boundaries. To this end, the sensor output is not expressed in terms of the cross-sectional state variables but is expressed in terms of the incoming and outgoing waves:

$$\tilde{y}(s) = \begin{bmatrix} \tilde{Y}_0(s) \\ 0 \end{bmatrix} w(0,s) + \begin{bmatrix} 0 \\ \tilde{Y}_1(s) \end{bmatrix} w(1,s) + \tilde{f}(s) = \tilde{Y}_{\text{out}}(s)w_{\text{out}}(s) + \tilde{Y}_{\text{in}}(s)w_{\text{in}}(s) + \tilde{f}(s) \tag{22}$$

where the matrix $\tilde{Y}_0(s) \in C^{q\times 2n}$ $(q \leq r)$ is assembled from rows of $Y(s)T(x_i,0,s)$, $\tilde{Y}_1(s) \in C^{(r-q)\times 2n}$ from rows of $Y(s)T(x_i,1,s)$, $\tilde{f}(s) \in C^r$ from rows of $Y(s)w_f(x_i,0,s)$ and $Y(s)w_f(x_i,1,s)$, respectively, and

$$\tilde{Y}_{\text{out}}(s) = \begin{bmatrix} \tilde{Y}_0(s)T_3 \\ \tilde{Y}_1(s)T_2 \end{bmatrix}, \qquad \tilde{Y}_{\text{in}}(s) = \begin{bmatrix} \tilde{Y}_0(s)T_1 \\ \tilde{Y}_1(s)T_4 \end{bmatrix} \tag{23}$$

The vector $\tilde{f}(s)$ represents contribution to the sensor outputs of disturbances between the boundaries $(x = 0,1)$ and the sensor positions. The closed-loop response at boundaries are obtained from Eqs. (17), (19), and (22) as

$$w_{\text{out}}(s) = S_{cl}(s)w_{\text{in}}(s) + R(s)C(s)\tilde{f}(s) \tag{24}$$

where the matrix $S_{cl}(s)$ is the closed-loop scattering matrix, and

$$S_{cl}(s) = \left[I_{2n} - R(s)C(s)\tilde{Y}_{\text{out}}(s)\right]^{-1} \times \left[S(s) + R(s)C(s)\tilde{Y}_{\text{in}}(s)\right] \tag{25}$$

It is shown in Eq. (24) that the outgoing waves are generated from reflection or transmission of the incoming waves through the closed-loop scattering matrix $S_{cl}(s)$ and from contribution to the sensor outputs of disturbances between the boundaries $(x = 0,1)$ and the sensor positions. Elements of the closed-loop scattering matrix $S_{cl}(s)$ are assignable freely to be zero through arrangement of elements of the gain matrix $C(s)$. Reflection/transmission of the jth component of the incoming waves to the ith component of the outgoing waves is eliminated by setting (i,j) element of $S_{cl}(s)$ zero. In special cases where the dimensions of $\tilde{y}(s)$ and $u(s)$ are equal to $2n$ and the matrices $R(s)$ and $\tilde{Y}_{\text{in}}(s)$ are nonsingular, the gain matrix to achieve $S_{cl}(s) = 0$ is simply obtained from Eqs. (18) and (25) as

$$C(s) = -R(s)^{-1}S(s)\tilde{Y}_{\text{in}}^{-1}(s) = B(s)^{-1}B_{\text{in}}(s)\tilde{Y}_{\text{in}}^{-1}(s) \tag{26}$$

The second term in Eq. (24) does not disappear unless the gain matrix $C(s)$ is a null matrix or the vector $\tilde{f}(s)$ is equal to a null vector. That is, the control effort cancels reflection/transmission from the incoming waves to the outgoing waves; however, it generates outgoing waves at the same time, when $\tilde{f}(s)$ is not a null vector. It is preferable to set the vector $\tilde{f}(s)$ a null vector to avoid influence of the disturbances between the boundaries and the sensor positions to the sensor outputs. This elimination of $\tilde{f}(s)$ is achieved by appropriately choosing sensor positions and the parameter q, which specifies the number of sensor outputs expressed by the wave modes $w(0,s)$ in Eq. (22). Because of the sparse structure of the matrix $Y(s)$ and the vector $f(x,s)$, rows of $Y(s)w_f(x_i,0,s)$ and $Y(s)w_f(x_i,1,s)$ associated with $\tilde{f}(s)$ can be set null by choosing sensor positions x_i and the parameter q appropriately, when disturbances distribution is restricted to be placed for several parts of the structure. The physical meaning of relation between the vector $\tilde{f}(s)$, sensor positions, and the parameter q is intuitively illustrated in Fig. 2. The vector $\tilde{f}(s)$ is a null vector if the disturbance does not occur between sensor positions and positions of actuators driven in response to the sensor outputs.

Controller Design with Model Reduction

When a structural network is partitioned into two subsystems, i.e., a subsystem with external disturbances and a subsystem without external disturbances, the dimension of the equation of the system can be reduced to the dimension of the subsystem with external disturbances. In such a case, Eqs. (6) and (7) can be arranged as

$$\frac{d}{dx}\begin{bmatrix} y_m(x,s) \\ y_s(x,s) \end{bmatrix} = \begin{bmatrix} A_m(s) & 0 \\ 0 & A_s(s) \end{bmatrix}\begin{bmatrix} y_m(x,s) \\ y_s(x,s) \end{bmatrix} + \begin{bmatrix} f_m(x,s) \\ 0 \end{bmatrix} \tag{27a}$$

$$\begin{bmatrix} B_{01}(s) & B_{02}(s) \\ B_{03}(s) & B_{04}(s) \end{bmatrix}\begin{bmatrix} y_m(x,s) \\ y_s(x,s) \end{bmatrix} + \begin{bmatrix} B_{11}(s) & B_{12}(s) \\ B_{13}(s) & B_{14}(s) \end{bmatrix}\begin{bmatrix} y_m(x,s) \\ y_s(x,s) \end{bmatrix} = \begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} u(s) \tag{27b}$$

where the vector $y_m(x,s)$ denotes the cross-sectional state vector of the subsystem with external disturbances, and the vector $y_s(x,s)$ the cross-sectional state vector of the subsystem without external disturbances. With inclusion of the dynamics of the subsystem $y_s(x,s)$ to the boundary conditions, Eqs. (27) are reduced to

$$\frac{\mathrm{d}y_m(x,s)}{\mathrm{d}x} = \boldsymbol{A}_m(s)y_m(x,s) + f_m(x,s) \tag{28a}$$

$$\boldsymbol{B}_{0m}(s)y_m(0,s) + \boldsymbol{B}_{1m}(s)y_m(1,s) = \boldsymbol{B}_m(s)\boldsymbol{u}(s) \tag{28b}$$

where

$$\boldsymbol{B}_{0m}(s) = \boldsymbol{B}_{01}(s) - \boldsymbol{D}_2(s)\boldsymbol{D}_4(s)^{-1}\boldsymbol{B}_{03}(s) \tag{28c}$$

$$\boldsymbol{B}_{1m}(s) = \boldsymbol{B}_{11}(s) - \boldsymbol{D}_2(s)\boldsymbol{D}_4(s)^{-1}\boldsymbol{B}_{13}(s) \tag{28d}$$

$$\boldsymbol{B}_m(s) = \boldsymbol{B}_1(s) - \boldsymbol{D}_2(s)\boldsymbol{D}_4(s)^{-1}\boldsymbol{B}_2(s) \tag{28e}$$

$$\boldsymbol{D}_2(s) = \boldsymbol{B}_{02}(s) + \boldsymbol{B}_{12}(s)e^{As(s)}$$

$$\boldsymbol{D}_4(s) = \boldsymbol{B}_{04}(s) + \boldsymbol{B}_{14}(s)e^{As(s)} \tag{28f}$$

The dimension of the state vector $y_m(x,s)$ in Eqs. (28) is less than the dimension of the original system in Eqs. (27). The reduction of dimension of the state vector leads to the scattering matrix of reduced dimension and simplifies manipulation required in contoller design. The model reduction technique presented here is necessary to design a controller canceling traveling waves that pass through the controller boundary at the beginning, reflect at different boundaries from the controlled boundary, and then return to the controlled boundary. In a design problem of a controller to cancel reflection of traveling waves at a midpoint of a member, the solution is that no control effort is required since there is no reflection of waves at a midpoint of a member by nature. However, applying the model reduction technique, the midpoint to be controlled is transformed to an endpoint with boundary conditions, including dynamics of the rest of the member, and a controller to cancel reflection of waves at the controlled point can be designed. An illustrative example will be presented in Sec. IV.

Controller Design with Local Models

The preceding discussions proceed with focus on global dynamics of structural networks rather than local dynamics. One of the advantages of the traveling wave approach is that a controller can be designed using only a local model of the controlled portion without requiring an exact model of other portions of the structure. The previous works[2-9,11] on traveling wave control study the problem from such a point of view. The general formulation of the local scattering behavior in structures is presented in Ref. 3. The essence of the formulation is that the boundary conditions at each boundary are expressed independently of other boundaries as

$$_k w_{\mathrm{out}}(s) = {}_k S(s)\, {}_k w_{\mathrm{in}}(s) + {}_k\boldsymbol{R}(s)\, {}_k\boldsymbol{u}(s)$$

$$(k = 1,2,\ldots,\nu; \quad {}_k w_{\mathrm{out}}, \quad {}_k w_{\mathrm{in}} \in \boldsymbol{C}^{n_k}) \tag{29a}$$

$$\sum_{j=1}^{\nu} n_k = 2n \tag{29b}$$

where ${}_k w_{\mathrm{out}}(s)$, ${}_k w_{\mathrm{in}}(s)$, ${}_k\boldsymbol{u}(s)$, and ${}_k\boldsymbol{S}(s)$ are outgoing waves, incoming waves, control forces, and scattering matrix associated with the boundary being considered, respectively. Controllers can be designed one by one for each boundary through use of Eqs. (29), and such a design leads to local (decentralized) control[17] in which only local state information is used to synthesize the control law for each actuator.

IV. Numerical Examples

Lateral Vibration of a Free-Free Flexible Beam

Lateral vibration of a free-free flexible beam is treated to demonstrate the wave-absorbing control with noncollocated sensors and actuators. A control torque $M_c(t)$ and a control force $V_c(t)$ are applied at one end of the beam. The beam is assumed to be a Bernoulli-Euler beam. The lateral vibration of the beam is described by the following partial differential equation neglecting damping and the higher order terms:

$$\rho\frac{\partial^2 y(x,t)}{\partial t^2} + EI\frac{\partial^4 y(x,t)}{\partial x^4} = 0 \tag{30}$$

with boundary conditions

$$EI\frac{\partial^2 y(x,t)}{\partial x^2}(0,t) = M_c(t), \qquad EI\frac{\partial^3 y(x,t)}{\partial x^3}(0,t) = V_c(t) \tag{31a}$$

$$EI\frac{\partial^2 y(x,t)}{\partial x^2}(l,t) = 0, \qquad EI\frac{\partial^3 y(x,t)}{\partial x^3}(l,t) = 0 \tag{31b}$$

where ρ, EI, and l denote the mass per unit length, the bending rigidity, and the total length of the beam, respectively. The external disturbances may be neglected to facilitate derivation of the control algorithm.

The equation of motion, Eq. (30), and the boundary conditions in Eq. (31) are Laplace transformed and expressed in the matrix form with the cross-sectional state vector $y(x,s) = [y(x,s),\ \theta(x,s),\ M(x,s),\ V(x,s)]^T$ to analyze the vibration of the beam in terms of traveling waves. The parameters $y(x,s)$, $\theta(x,s) = \mathrm{d}y(x,s)/\mathrm{d}x$, $M(x,s) = EI[\mathrm{d}^2y(x,s)/\mathrm{d}x^2]$, and $V(x,s) = EI[\mathrm{d}^3y(x,s)/\mathrm{d}x^3]$ are the Laplace transforms of each parameter, i.e., the deflection, the slope, the internal bending moment, and the internal shear force of the beam, respectively. The matrices corresponding to $\boldsymbol{A}(s)$, $\boldsymbol{B}_0(s)$, $\boldsymbol{B}_1(s)$, $\boldsymbol{B}(s)$, and $\boldsymbol{u}(s)$ in Eqs. (6) and (7) are given for this case as

$$\boldsymbol{A}(s) = \begin{bmatrix} 0 & l & 0 & 0 \\ 0 & 0 & l/EI & 0 \\ 0 & 0 & 0 & l \\ \lambda^4 EI & 0 & 0 & 0 \end{bmatrix}, \quad \lambda = \lambda(s) = \left(\frac{\rho}{EI}\right)^{1/4}\sqrt{-js} \tag{32a}$$

$$\boldsymbol{B}_0 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B}_0 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_2 \end{bmatrix} \tag{32b}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{I}_2 \\ \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{u}(s) = \begin{bmatrix} M_c(s) \\ V_c(s) \end{bmatrix} \tag{32c}$$

It is apparent from Eq. (31) or (32) that there is no interconnection between two ends of the beam, and the controller can be designed to cancel outgoing waves at one end $x = 0$ without consulting the boundary conditions at another end $x = l$. The diagonalization of the equation is carried out with the following matrix:

$$Y(s) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ j\lambda & \lambda & -j\lambda & -\lambda \\ -M_0 & M_0 & -M_0 & M_0 \\ -j\lambda M_0 & \lambda M_0 & j\lambda M_0 & -\lambda M_0 \end{bmatrix}$$

$$M_0 = M_0(s) = \lambda^2(s)EI \tag{33}$$

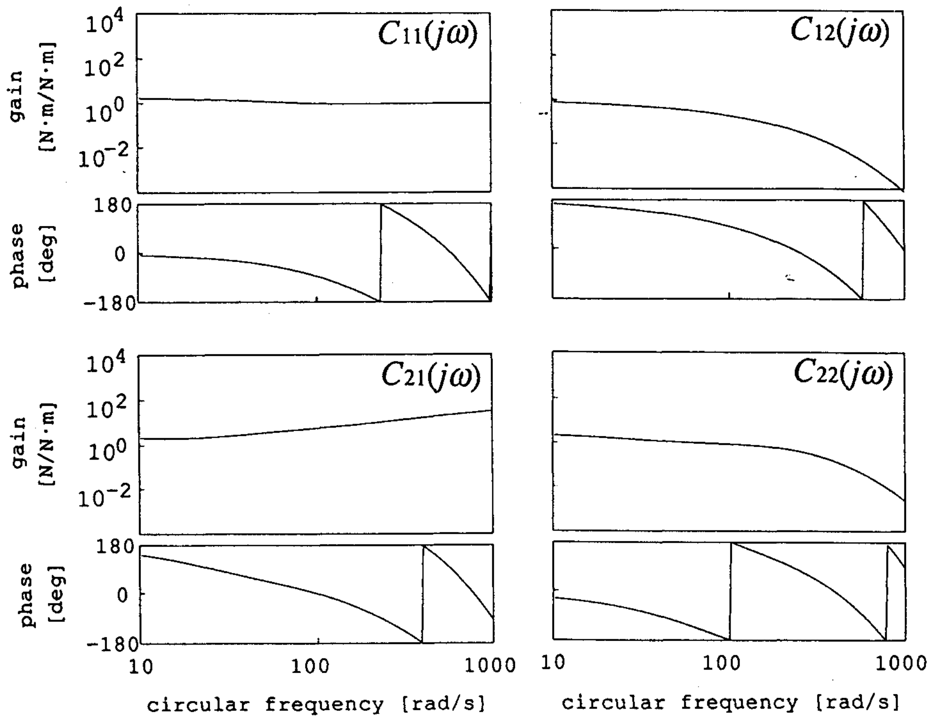


Fig. 3 Bode plots of elements of the gain matrix $C(j\omega)$ for the free-free beam.

The diagonalization yields the diagonalized matrix $\Lambda(s)$ and the transfer matrix of the wave modes $T(x,x_0,s)$ as follows:

$$\Lambda(s) = \operatorname{diag}(j\lambda,\ \lambda,\ -j\lambda,\ -\lambda) \tag{34a}$$

$$T(x,x_0,s) = e^{\Lambda(s)}$$

$$= \operatorname{diag}\left[e^{j\lambda(x-x_0)},\ e^{\lambda(x-x_0)},\ e^{-j\lambda(x-x_0)},\ e^{-\lambda(x-x_0)}\right] \tag{34b}$$

The boundary conditions at $x=0$ are expressed in terms of relations between outgoing waves, incoming waves, and the control forces as

$$b(0,s) = S_0(s)a(0,s) + R_0(s)u(s) \tag{35a}$$

where

$$S_0(s) = \begin{bmatrix} -j & 1+j \\ 1-j & j \end{bmatrix},\quad R_0(s) = -\frac{1+j}{2\lambda M_0}\begin{bmatrix} -j & 1+j \\ 1-j & j \end{bmatrix} \tag{35b}$$

A controller that has sensors of the bending moment at two points on the beam, $x=l_1$ and $x=l_2$, is designed here to demonstrate the wave-absorbing control with noncollocated sensors and actuators. The controller has the following form:

$$u(s) = C(s)\begin{bmatrix} M(l_1,s) \\ M(l_2,s) \end{bmatrix} \tag{36}$$

The sensor measurements are expressed in terms of the wave modes at the end $x=0$:

$$\begin{bmatrix} M(l_1,s) \\ M(l_2,s) \end{bmatrix} = M_0\begin{bmatrix} -e^{j\lambda l_1} & e^{\lambda l_1} \\ -e^{j\lambda l_2} & e^{\lambda l_2} \end{bmatrix} a(0,s)$$

$$+ M_0\begin{bmatrix} -e^{-j\lambda l_1} & e^{-\lambda l_1} \\ -e^{-j\lambda l_2} & e^{-\lambda l_2} \end{bmatrix} b(0,s) \tag{37}$$

The gain matrix $C(s)$ is selected from Eqs. (35), (36), and (37) to set the closed-loop scattering matrix $S_{cl}(s)$ a null matrix as

$$C(s) = \frac{1}{e^{\lambda(l_1+jl_2)} - e^{\lambda(l_2+jl_1)}} \times \begin{bmatrix} e^{j\lambda l_2} - e^{\lambda l_2} & e^{\lambda l_1} - e^{j\lambda l_1} \\ \lambda(e^{j\lambda l_2} - je^{\lambda l_2}) & \lambda(je^{\lambda l_1} - e^{j\lambda l_1}) \end{bmatrix} \tag{38}$$

Each element of the gain matrix is analytic in the open right half of the complex plane and causal.

The open-loop and closed-loop transfer functions from a point force at $x=l_f$ to the tip deflection of the beam are analyzed numerically to clarify improvement of the structural response due to the disturbances. The beam parameters are chosen as the unit length, mass, and the bending rigidity, and viscous damping is included as $c=0.5$. The sensor positions l_1/l and l_2/l are set to 0.3 and 0.6, respectively. Bode plots of (i,j)th element $C_{ij}(j\omega)$ of the matrix $C(j\omega)$ are shown in Fig. 3. The elements $C_{11}(j\omega)$ and $C_{21}(j\omega)$ have infinite bandwidth, and $C_{12}(j\omega)$ and $C_{22}(j\omega)$ have finite bandwidth. The phase of each element lags approximately in proportional to $\sqrt{\omega}$. This is due to the traveling time of the disturbances from the sensor positions to the actuator position. The traveling time depends upon the frequency of waves in such a dispersive medium as the Bernoulli-Euler beam. The finiteness of the bandwidth and those phase characteristics have advantages to realize the controller through use of analog circuits or digital computers.

Open- and closed-loop responses of the tip deflection to a point force are compared in Figs. 4 and 5 to clarify the performance of the controller. The responses are calculated employing exact solutions, Eq. (11) or (21), of the equation of motion including external force term. The closed-loop responses are calculated for some locations of the excitation point. The open- and closed-loop response ($l_f/l=0.8$) is shown in Fig. 4. Both sensors are located between positions of the excitation point and the actuator in the case shown in the figure. The excited disturbances arrive at the actuator position after they pass through two sensor positions in

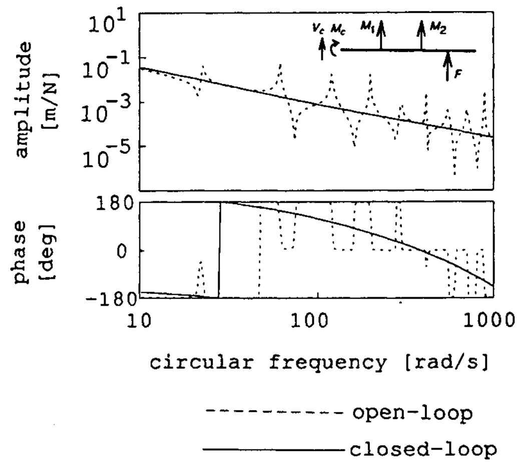


Fig. 4 Open- and closed-loop response of the tip deflection ($l_f/l = 0.8$).

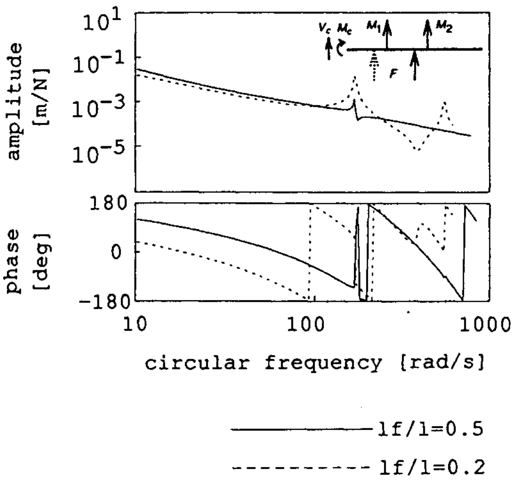


Fig. 5 Closed-loop responses of the tip deflection of the beam ($l_f/l = 0.5$, 0.2).

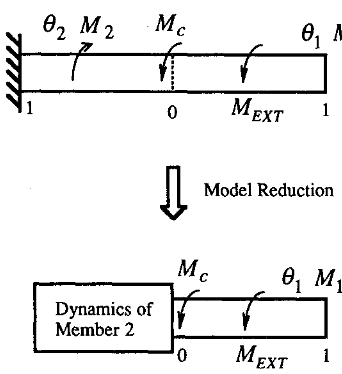


Fig. 6 Fixed-free bar.

this case. All peaks of amplitude of the response are seen to be removed by the control as is shown in this figure. The beam is excited between the two positions or at a location closer to the actuator position than the two sensor positions in cases shown in Fig. 5. It is observed in the figure that an unstable pole appears. Poles at the same frequency for the two cases are the identical pole of $[\boldsymbol{B}_0(s)+\boldsymbol{B}_1(s)\phi(s)-\boldsymbol{B}(s)\boldsymbol{C}(s)\tilde{\phi}(s)]^{-1}$, although one pole looks as if it were in the right half plane due to mechanical plotting of the phase. The isolated pole at about 560 rad/s will be easily seen to be unstable. In these cases, stability of the closed-loop is not guaranteed because of contribution to the sensor outputs of disturbances between the actuator position and the sensor positions.

Torsional Vibration of a Fixed-Free Bar

This example treats torsional vibration of a bar with one end fixed and another end free (Fig. 6). An external control torque is exerted at the midpoint of the bar. Disturbances are assumed to occur between the free end and the actuator position. The controller is designed to cancel traveling waves passing through the actuator position after its reflection at the fixed end. The present system is expressed with the following vectors and matrices:

$$y(s) = [\theta_1(s) \quad M_1(s) \quad \theta_2(s) \quad M_2(s)]^T \tag{39a}$$

$$\tilde{f}(s) = [0 \quad M_f(s)\delta(x-l_f) \quad 0 \quad 0]^T \tag{39b}$$

$$A(s) = \begin{bmatrix} A_1(s) & \mathbf{0} \\ \mathbf{0} & A_2(s) \end{bmatrix}, \quad A_i(s) = \begin{bmatrix} 0 & l_i/GJ_i \\ s^2 l_i I_i & 0 \end{bmatrix} \quad (i=1,2) \tag{39c}$$

$$\boldsymbol{B}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{39d}$$

The bar is partitioned into two parts in the formulation, and the actuator position is treated as a junction. The present system is reduced to a system with the state vector of two dimensions employing the technique of model reduction presented in the preceding section:

$$y(s) = [\theta_1(s) \quad M_1(s)]^T, \quad \tilde{f}(s) = [0 \quad M_f(s)\delta(x-l_f)]^T$$

$$A(s) = A_1(s) \tag{40a}$$

$$\boldsymbol{B}_0 = \begin{bmatrix} 0 & 0 \\ \lambda_2 GJ_2 \coth\lambda_2 l_2 & -1 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{40b}$$

where

$$\lambda_i = \lambda_i(s) = s\sqrt{I_i/GJ_i}, \qquad (i=1,2) \tag{40c}$$

Dynamics of the portion between the fixed end and the actuator location is included in the boundary condition in Eqs. (40). Now, the actuator position is regarded as an endpoint of the bar, although it is actually a midpoint of the bar. Expression in Eqs. (40) enables one to design wave-absorbing controller that cancels reflection at the actuator position, while reflection of traveling waves at the actuator position does not appear if the expression in Eqs. (39) is used directly. The cross-sectional state vector $y(s)$ is transformed into the wave modes by the matrix

$$Y(s) = \begin{bmatrix} 1 & 1 \\ \lambda_1 GJ_1 & -\lambda_1 GJ_1 \end{bmatrix} \tag{41}$$

The controller of the following form is designed here:

$$M_c(s) = C(s)\theta_1(l_s, s) \tag{42}$$

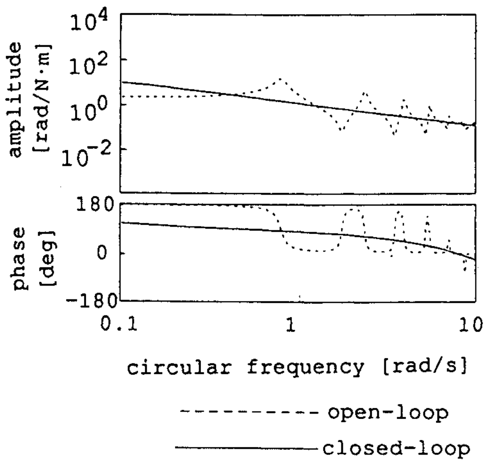
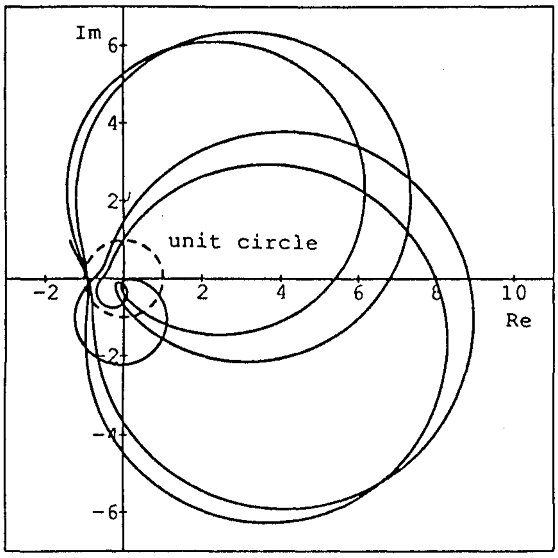
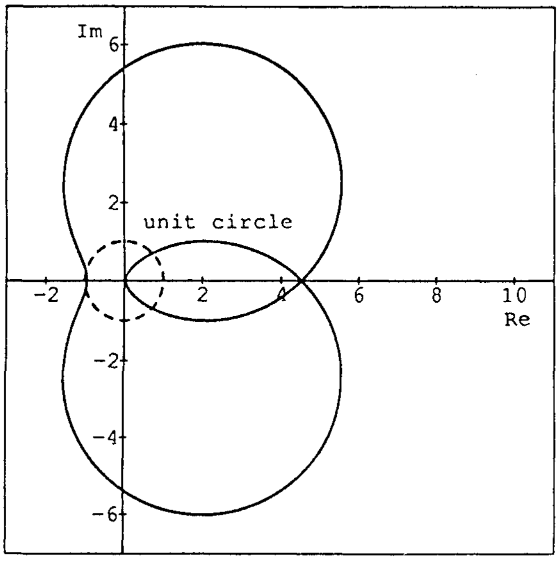


Fig. 7 Open- and closed-loop responses of the tip angle of the bar ($l_f/l_1=0.8$).



a) $l_s/l_1=0.3$ (noncollocation)



b) $l_s/l_1=0$ (collocation)

Fig. 8 Nyquist plots for the fixed-free bar.

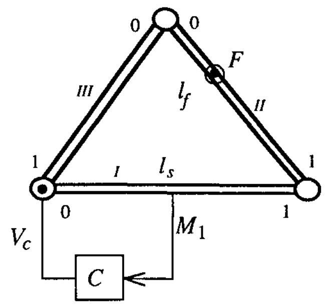


Fig. 9 Triangular truss.

The open-loop and closed-loop scattering behavior are obtained as follows:

$$b(0,s) = -\frac{\lambda_2 GJ_2 \coth\lambda_2 l_2 - \lambda_1 GJ_1}{\lambda_2 GJ_2 \coth\lambda_2 l_2 + \lambda_1 GJ_1} a(0,s) + \frac{M_c(s)}{\lambda_2 GJ_2 \coth\lambda_2 l_2 + \lambda_1 GJ_1} \tag{43}$$

$$b(0,s) = -\frac{\lambda_2 GJ_2 \coth\lambda_2 l_2 - \lambda_1 GJ_1 - C(s)e^{\lambda_1 l_s}}{\lambda_2 GJ_2 \coth\lambda_2 l_2 + \lambda_1 GJ_1 - C(s)e^{\lambda_1 l_s}} a(0,s) \tag{44}$$

The gain is determined to cancel the outgoing waves as

$$C(s) = (\lambda_2 GJ_2 \coth\lambda_2 l_2 + \lambda_1 GJ_1)e^{-\lambda_1 l_s} \tag{45}$$

The open- and closed-loop responses of the tip angle $\theta_1(l_1,s)$ to the excitation torque at $l_f/l_1=0.8$ are shown in Fig. 7. Parameters are chosen as unit values, the sensor is located at $l_s/l_1=0.3$, and viscous damping is included as $c_i=0.1$ in the calculation. A pole is at the origin of the complex plane in the closed-loop response, i.e., the wave-absorbing control does not stabilize the rigid motion.

The present system is a single-input/single-output (SISO) system, and evaluation of the classical gain and phase margin is possible. The Nyquist plots of the loop transfer functions for the present system with a sensor at $l_s/l_1=0.3$ (noncollocation) and with a sensor at $l_s/l_1=0$ (collocation) is shown in Fig. 8. The gain margin and the phase margin for noncollocation are equal to 0.63 dB and 14 deg, respectively. The gain margin and the phase margin for collocation are equal to 0.47 dB and 12 deg, respectively. Collocation of sensor and actuator does not improve the stability margin for the present system compared with the case with the noncollocated sensor and actuator. In both cases of noncollocaiton and collocation, since the gain and phase margin are small, modeling errors and such a roll-off as accompanied by a large phase lag may cause instability easily.

Triangular Truss

The last example treats an equilateral triangular truss (Fig. 9). The truss is assumed to be constructed from three uniform Bernoulli-Euler beams that are connected to each other by ideal joints transmitting only displacements and forces. An external control force is exerted at a joint. The controller of the form

$$V_c(s) = C(s)M_1(l_s,s) \tag{46}$$

is designed here to cancel transmission of a traveling wave mode from the member I to the member III. The controller is designed by using the local model at the controlled joint. The open- and closed-loop responses of the displacement of the member I at $x/l_1=0.5$ to an excitation at $l_f/l_2=0.2$ on the member II are shown in Fig. 10. Again, parameters are chosen as unit values, the sensor is located at $l_s/l_1=0.3$, and viscous

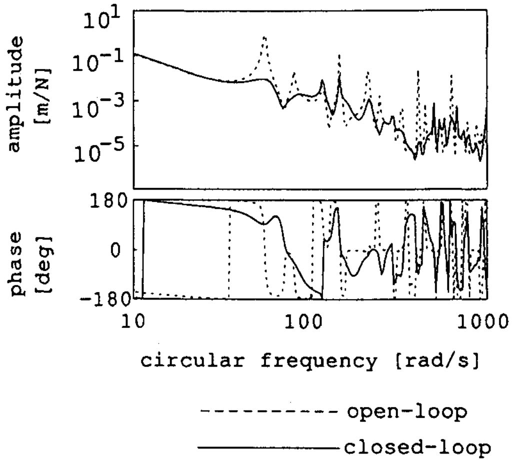


Fig. 10 Open- and closed-loop responses of the displacement of the member I at $x/l_1 = 0.5$ ($l_f/l_2 = 0.2$).

damping is included as $c_i = 0.5$ in the calculation. Improvement of the response in the closed-loop is seen partially since the controller is too simple to control many wave modes existing in the structure.

V. Conclusions

Design concepts of wave-absorbing controllers with noncollocated sensors and actuators are introduced in this paper. Partial differential equations are employed as the mathematical model of a structural network for the controller design. The Laplace transform of the equations with respect to time and introduction of a cross-sectional state vector yield a first-order ordinary differential equation in the matrix form to express vibration of structural members in terms of wave modes. Amplitude of a wave mode at any point on the structure is related with that at another point by a transfer matrix. The control algorithm for the noncollocated controller is obtained through use of such a transfer matrix.

Numerical examples show that the noncollocated controller works quite well when the excitation does not occur between the positions of the sensors and the position of the actuator. The controller with enough information to cancel the outgoing waves at the end of the beam can be obtained by locating sensors so as to measure propagating disturbances downstream of the excited position. It will be seen to be natural that the control does not guarantee stability if members are excited between sensor positions and actuator positions. The stability margins of the controller are also examined for an SISO case, and the result shows little robustness of the controller for modeling errors or roll-off.

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